Refined Engineering Beam Theory Based on the Asymptotic Expansion Approach

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A systematic derivation of a refined engineering theory governing the response of elastic beams is presented. The method employed to accomplish this is that of asymptotic expansion that combines dimensional analysis with the expansion in powers of a small parameter of the solution of the three-dimensional linear elasticity theory. The present beam theory contains more information than the classical Timoshenko theory. A new shear coefficient is defined and compared with existing ones.

Nomenclature

A = dimensionless area of cross section of beam
D = domain of beam
d = transverse displacement of beam axis
E = Young's modulus
I = dimensionless moment of inertia of cross section about x_2 axis
K = constant
k = shear coefficient
L = length of beam
l_a = tension coefficient
M = bending moment
N = axial force
n_i = outward unit normal vector
p = surface loading
Q = shear force
R = typical scale of beam cross section
S = domain of cross section
s = dimensionless arc length
$\bar{t}_i$ = surface traction
U = deflection of beam in engineering formulation
u_i = displacement component
$\bar{u}_i$ = prescribed displacement component
v_i = dimensionless displacement component
W = axial displacement of beam axis
x_i = rectangular Cartesian coordinate
δ_{ij} = Kronecker delta
∂D = boundary of D
∂S = boundary of S
ϵ = small-beam parameter
ν = Poisson's ratio
σ = representative stress level
σ_{ij} = stress tensor component
ϕ = rotation of beam
τ_{ij} = dimensionless stress tensor component
χ = auxiliary function

I. Introduction

IN the present paper, an engineering theory for homogeneous, isotropic beams based on the application of the asymptotic technique to the equations of three-dimensional elasticity theory is derived. The well-known Bernoulli-Euler[1] and Timoshenko[1] beam formulations are widely used in engineering calculations. The latter theory is employed when the traverse shear deformation cannot be neglected. Although the Timoshenko theory does not improve the elementary theory to the second asymptotic order, it does capture part of the correction inherent in the second order. The presence of the well-known shear coefficient, however, has been a concern of researchers for a long time. Also, different definitions of the coefficient have been proposed. Some researchers, including Timoshenko, defined this coefficient by matching a natural frequency for the free vibration of a beam obtained through use of the Timoshenko theory to that from the exact three-dimensional elasticity theory. This approach has suffered due to the general paucity of exact solutions for complex shape cross sections. Another group of researchers define this coefficient using elastostatic considerations only. A representative work along this line was contributed by Cowper.[2] Without any dependence on the corresponding exact solution, he derived many values of k (shear coefficient) for various cross-section shapes. A fact that should be noticed is that Cowper's k does not coincide with the k obtained by the first group of authors. Recent comparisons by Hutchinson[3,4] demonstrated that some confusion exists in the selection of the correct shear coefficient.

The development of a more accurate refined beam theory has been an active topic for research for the last three decades. For example, Dokmeci[5] expanded the displacements in terms of complete polynomials; all the control equations and corresponding boundary conditions were then obtained via a variational principle. Recently, Rehfield and Murthy[6] considered a plane stress beam. By making use of the exact elasticity solution, they introduced some corrections to the Timoshenko theory. Rychter[7] provided error estimates for this theory. It and all other refined theories are classified as engineering theories since they do not include an analysis of the boundary layers. It is therefore difficult to estimate the error introduced by the assumptions forming the basis of these theories. A more rigorous approach is needed to provide some benchmark solutions. Recent papers by Widera et al.[8] on the analysis of pipes, by Duva and Simmonds,[9] and by Wan[10] are examples of work in this area.

Here, we develop a linear, elastic engineering beam theory that is based on the asymptotic expansion technique.[11] Full asymptotic expansions (inner and outer) are difficult to employ in engineering applications because the development of boundary-layer solutions is too complex a problem for them to be considered. Therefore, only the outer expansion is devel-

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oped in Sec. II. This expansion is then converted in Sec. III to an engineering theory by introducing some engineering variables, such as the deflection of the centerline and the bending moment. The boundary conditions are also expressed in terms of these variables. On comparison with beam theories derived from variational principles, it is noted that using the present theory makes it easier to determine where the error is introduced. In particular, when comparing it to the Timoshenko theory, the present formulation is seen to capture more information from the outer expansion while adding only a few extra calculations. A new shear coefficient is defined and compared with Cowper's definition.

II. Asymptotic Expansion

Asymptotic expansions in general consist of an outer expansion that is valid in the interior of the beam and an inner expansion that is valid near the ends. The latter expansion is too complex to be considered for engineering calculations and will thus not be discussed in the present work. The outer expansion is straightfoward but tedious. The procedure is therefore only briefly outlined in this paper. Readers are referred to Fan's work[12] for the detailed and complete derivation.

The equations of linear elasticity theory in tensor form are written as[13]

$$\sigma_{ij,j} = 0 \quad \text{in } D \tag{1a}$$

$$(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} = (E/2)(u_{i,j} + u_{j,i}) \tag{1b}$$

with the following boundary conditions

$$\sigma_{ij}n_j = \bar{t}_i \quad \text{on } \partial D_t \tag{2a}$$

$$u_i = \bar{u}_i \quad \text{on } \partial D_u \tag{2b}$$

where $(\ )_{,i} = \partial(\ )/\partial y_i$. Here, D is the domain of the elastic beam, ∂D_t the boundary for the prescribed stresses, and ∂D_u the corresponding one for prescribed displacements. Repeated indices imply the use of the summation convention. The coordinates are shown in Fig. 1a. Our formulation is valid for arbitrary shape cross sections. The elliptical cross section shown in Fig. 1b is used for the numerical calculations carried out in Sec. IV.

Dimensionless variables are now introduced in the following manner:

$$X_\alpha = y_\alpha/R, \qquad x_3 = y_3/L \tag{3a}$$

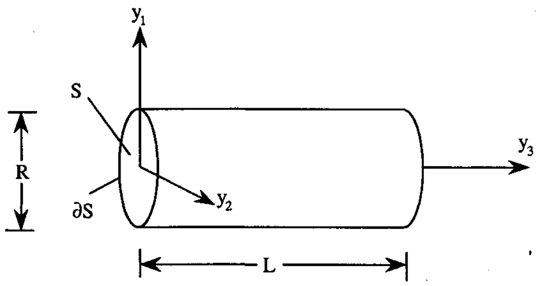

Fig. 1a Coordinates and dimensions.

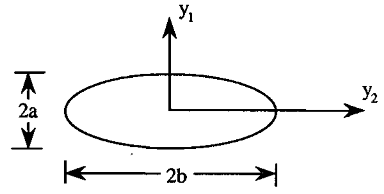

Fig. 1b Elliptical cross section.

$$\tau_{ij} = \sigma_{ij}/\sigma, \qquad v_i = [u_i(E/L)\sigma] \tag{3b}$$

where R and L are shown in Fig. 1, and σ is a typical stress level. Note that x_3 is the dimensionless axial coordinate while x_1, x_2 are the cross-section coordinates. With these dimensionless variables, Eqs. (1) can be rewritten as:

$$\tau_{i\alpha,\alpha} + \epsilon\tau_{i3,3} = 0 \tag{4a}$$

$$(1+\nu)\tau_{\alpha\beta} - \nu\tau_{kk}\delta_{\alpha\beta} = \tfrac{1}{2}\epsilon(v_{\alpha\beta} + v_{\beta,\alpha}) \tag{4b}$$

$$\tau_{3\alpha} = \tfrac{1}{2}[v_{\alpha,3} + (1/\epsilon)v_{3,\alpha}] \tag{4c}$$

$$\tau_{33} - \nu\tau_{\alpha\alpha} = v_{3,3} \tag{4d}$$

where $\epsilon = R/L \ll 1$. Hereafter $(\ )_{,i} = \partial(\ )/\partial x_i$. Also, Latin letter subscripts i, j, etc., take on the values of 1, 2, and 3 while Greek leters, α, β, etc., represent 1 and 2.

The expansions for the stresses and displacements of symmetric bending are given as follows:

$$v_\alpha \sim \sum_{n=0}^{\infty} \epsilon^n v_\alpha^{(n)}$$

$$v_3 \sim \epsilon \sum_{n=0}^{\infty} \epsilon^n v_3^{(n)}$$

$$\tau_{\alpha\beta} \sim \epsilon^3 \sum_{n=0}^{\infty} \epsilon^n \tau_{\alpha\beta}^{(n)}$$

$$\tau_{3\alpha} \sim \epsilon^2 \sum_{n=0}^{\infty} \epsilon^n \tau_{3\alpha}^{(n)}$$

$$\tau_{33} \sim \epsilon \sum_{n=0}^{\infty} \epsilon^n \tau_{33}^{(n)} \tag{5}$$

It is to be noted that expansions (5) do not presuppose any relative order of magnitude. The only imposed requirement is tht we wish to restrict our attention to bending deformations. A detailed examination indicates that 1) the terms of $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ in $\tau_{\alpha\beta}$ correspond to plane strain deformation, 2) the terms of $\mathcal{O}(1)$ in τ_{33} and v_3 represent uniaxial tension in the x_3 direction, and 3) the $\mathcal{O}(1)$ terms in $\tau_{3\alpha}$ are associated with torsion about the x_3 axis. Since our focus is on bending, these terms are not included in the expansions.

Upon substituting expansions (5) into the equilibrium and Hooke's law equations (4), one obtains upon equating terms of the same power in ϵ sequences of systems of differential equations. The equations for each sequence can be integrated with respect to x_1 and x_2 in a step-by-step manner. The first two-order approximations for the displacements are

$$v_1 = d^{(0)}(x_3) + \epsilon^2\left[\frac{\nu}{2}(x_1^2 - x_2^2)d_{,33}^{(0)}(x_3) + d^{(2)}(x_3)\right] \tag{6a}$$

$$v_2 = \epsilon^2 \nu x_1 x_2 d_{,33}^{(0)}(x_3) \tag{6b}$$

$$v_3 = \epsilon\{-x_1 d_{,3}^{(0)}(x_3) + \epsilon^2[\chi(x_1,x_2)d_{,333}^{(0)}(x_3) - x_1 d_{,3}^{(2)}(x_3) + V_3(x_3)]\} \tag{6c}$$

where d is the transverse displacement of the beam axis and is determined from

$$I\, d_{,3333}^{(0)}(x_3) = p, \qquad d_{,3333}^{(2)}(x_3) = 0 \tag{7}$$

Here,

$$I = \int_S x_1^2\, dA, \qquad p(x_3) = \int_{\partial S} \tau_{1\alpha}^{(0)} n_\alpha ds \tag{8}$$

and the auxiliary function $\chi(x_1,x_2)$ is defined in Appendix A. A simplification regarding the shape of the cross section was made in the preceding by assuming the x_1 axis to be an axis of symmetry.

The stress components in the cross section are defined by following boundary value problems:

$$\tau^{(0)}_{\alpha\beta,\beta} = -\tau^{(0)}_{3\alpha,3} \tag{9a}$$

$$(1+\nu)\tau^{(0)}_{\alpha\beta} - \nu[\tau^{(0)}_{\lambda\lambda} + \tau^{(2)}_{33}]\delta_{\alpha\beta} = \tfrac{1}{2}[v^{(4)}_{\alpha,\beta} + v^{(4)}_{\beta,\alpha}] \tag{9b}$$

$$\tau^{(0)}_{\alpha\beta} n_\alpha = \bar{t}_\beta \qquad \text{on } \partial S \tag{9c}$$

while the transverse shear stresses are given by

$$\tau^{(0)}_{31} = \frac{1}{2(1+\nu)}\left[\frac{\nu}{2}(x_1^2 - x_2^2) + \chi,_1\right] d,^{(0)}_{333}(x_3) \tag{10a}$$

$$\tau^{(0)}_{32} = \frac{1}{2(1+\nu)}(\nu x_1 x_2 + \chi,_2) d,^{(0)}_{333}(x_3) \tag{10b}$$

Details regarding the derivation of Eqs. (10a) and (10b) and the auxiliary function $\chi(x_1,x_2)$ can be found in any standard textbook on elasticity theory, such as Sokolnikoff.[13] The bending stress is given by

$$\tau^{(0)}_{33} = -x_1 d,^{(0)}_{33}(x_3) \tag{11a}$$

$$\tau^{(2)}_{33} = -x_1 d,^{(2)}_{33}(x_3) + \chi(x_1,x_2)\frac{p}{I} + \nu\tau^{(0)}_{\alpha\alpha} + V_{3,3}(x_3) \tag{11b}$$

The preceding expansion terms correspond to the outer expansion in the perturbation approach and are thus valid only in the interior of the beam. An inner expansion is needed to describe the deformation field in the zones near the end of the beam as well as to obtain the correct boundary conditions for the outer expansion. All engineering beam theories do not, however, consider this boundary-layer solution. In the following it will be our intent to develop an engineering beam theory based on the outer expansion only.

III. Refined Engineering Beam Theory

This beam formulation is mainly based on the previous outer asymptotic expansion. The theory thus has built into it a zeroth-order error, as do the Bernoulli and Timoshenko beam theories. In comparison with the latter, the present theory incorporates in it more information from the outer expansion. In addition to the transverse shear effect, the effect of distributed loading along the x_3 axis is also taken into account. The latter is of the same order as that for the shear deformation. The development of the refined theory starts with definitions of some engineering notation

$$U(x_3) = v_1(0,0,x_3) = d^{(0)}(x_3) + \epsilon^2 d^{(2)}(x_3) \tag{12}$$

$$W(x_3) = \frac{v_3(0,0,x_3)}{\epsilon^3} = \chi(0,0) d,^{(0)}_{333}(x_3) + V_3(x_3) \tag{13}$$

Equations (12) and (13) represent the axial and transverse displacements of the centerline of the beam. Note that W can be defined in other ways when the origin (0,0) is not in beam cross section S. The "rotation" can also be defined in several ways. We take note of some popular definitions here. With $\int_S x_\alpha \mathrm{d}A = 0$, we have

$$\varphi_1 = \frac{v_{3,1}(0,0,x_3)}{\epsilon} = -U(x_3) + \epsilon^2 \chi,_1(0,0)\frac{Q}{I} \tag{14a}$$

$$\varphi_2 = \frac{1}{\epsilon I}\int_S x_1 v_3 \mathrm{d}A = -U(x_3) + \epsilon^2 \frac{\int_S x_1 \chi \mathrm{d}A}{I} Q \tag{14b}$$

$$\varphi_3 = \frac{1}{\epsilon A}\int_S v_{3,1} \mathrm{d}A \tag{14c}$$

Stress resultants are defined as follows:

$$N = \int_S [\tau^{(0)}_{33} + \epsilon^2 \tau^{(2)}_{33}] \mathrm{d}A \tag{15a}$$

$$M = \int_S x_1 [\tau^{(0)}_{33} + \epsilon^2 \tau^{(2)}_{33}] \mathrm{d}A \tag{15b}$$

$$Q = \int_S [\tau^{(0)}_{31} + \epsilon^2 \tau^{(2)}_{31}] \mathrm{d}A \tag{15c}$$

It is easy to see that the equilibrium equations are given by

$$N,_3 = 0 \tag{16a}$$

$$M,_3 = Q \tag{16b}$$

$$Q,_3 = -p(x_3) \tag{16c}$$

Relations between the kinematic variables $U(x_3)$, $W(x_3)$, and $\varphi_i(x_3)$ and the resultant forces and moment Q, N, and M are established through the use of the outer expansion given in Sec. II. From Eqs. (11) and (15) we have

$$N = \epsilon^2\left\{A W,_3(x_3) + \frac{p}{I}\left[\int_S \chi \mathrm{d}A - A\chi(0,0)\right] + \nu\int_S \tau^{(0)}_{\alpha\alpha}\mathrm{d}A\right\} \tag{17a}$$

$$M = -IU,_{33}(x_3) + \epsilon^2\left\{\int_S x_1\left[\nu\tau^{(0)}_{\alpha\alpha} + \chi(x_1,x_2)\frac{p}{I}\right]\mathrm{d}A\right\} \tag{17b}$$

Note that $\tau^{(0)}_{\alpha\beta}$ are defined by Eqs. (9). The determination of these stress components constitutes a two-dimensional elasticity problem, which is not a desirable property for an engineering beam theory. Thus, we rewrite the integral involving $\tau^{(0)}_{\alpha\alpha}$ as follows:

$$\int_S \tau^{(0)}_{\alpha\alpha}\mathrm{d}A = \int_{\partial S} x_\alpha \tau^{(0)}_{\alpha\beta} n_\beta \mathrm{d}s - \int_S x_\alpha \tau^{(0)}_{\alpha,3}\mathrm{d}A \tag{18a}$$

$$\int_S x_1 \tau^{(0)}_{\alpha\alpha}\mathrm{d}A = \int_S [x_1 x_\alpha \tau^{(0)}_{\alpha 3,3} - \tfrac{1}{2} x_\alpha x_\alpha \tau^{(0)}_{31,3}]\mathrm{d}A + \int_{\partial S} [x_1 x_\alpha \tau^{(0)}_{\alpha\beta} n_\beta - \tfrac{1}{2} x_\alpha x_\alpha \tau^{(0)}_{1\beta} n_\beta]\mathrm{d}s \tag{18b}$$

where $\tau^{(0)}_{\alpha\beta} n_\beta = \bar{t}_\alpha$ are prescribed on ∂S, and $\tau^{(0)}_{3\alpha}$ are given by Eqs. (10). Thus,

$$N = \epsilon^2\left[A W,_3(x_3) - l_a \frac{p}{I} + \int_{\partial S} x_\alpha \bar{t}_\alpha \mathrm{d}s\right] \tag{19a}$$

$$M = -IU,_{33}(x_3) + \epsilon^2\left[\nu\int_{\partial S}(x_1 x_\alpha \bar{t}_\alpha - \tfrac{1}{2} x_\alpha x_\alpha \bar{t}_1)\mathrm{d}s + \frac{p}{I}k_a\right] \tag{19b}$$

where l_a is the tension coefficient,

$$l_a = \int_S\left(\frac{\nu}{2(1+\nu)}\left\{x_1\left[\frac{\nu}{2}(x_1^2 - x_2^2) + \chi,_1\right] + x_2(\nu x_1 x_2 + \chi,_2)\right\} - \chi\right)\mathrm{d}A + A\chi(0,0) \tag{20a}$$

and k_a is the shear coefficient and is defined as

$$k_a = \int_S\left(\frac{\nu}{2(1+\nu)}\left\{\tfrac{1}{2}(x_1^2 - x_2^2)\left[\frac{\nu}{2}(x_1^2 - x_2^2) + \chi,_1\right]\right.\right.$$

$$+ x_1x_2(\nu x_1x_2 + \chi_{,2})\Big\} + x_1\chi\Big)\mathrm{d}A \tag{20b}$$

The subscript a on l and k indicates that these coefficients were obtained via the asympototic expansion procedure.

Upon substitution of Eqs. (19) into Eqs. (16), we finally obtain

$$AW_{,33}(x_3) = \left(l_a \frac{p}{I} - \nu \int_{\partial S} x_\alpha \bar{t}_\alpha \mathrm{d}S\right)_{,3} \tag{21a}$$

$$IU_{,3333}(x_3) = p(x_3) + \epsilon^2\left(\nu \int_{\partial S} (x_1x_\alpha\bar{t}_\alpha - \tfrac{1}{2}x_\alpha x_\alpha \bar{t}_1) \times \mathrm{d}S + \frac{p}{I}k_a\right)_{,33} \tag{21b}$$

These equations, as well as one of Eqs. (14), and Eqs. (16) and (19) constitute our new engineering beam theory. The corresponding boundary conditions for Eqs. (21) at each end of the beam are expressed in terms of engineering displacements U, W, and φ_i or resultants Q, N, and M, as follows:

1) $U(x_3^0)$ or M, and W or N prescribed.
For example, M prescribed at $x_3 = x_3^0$ means

$$IU_{,33}(x_3^0) = -M(x_3^0) + \epsilon^2\left[\nu \int_{\partial S} (x_1x_\alpha\bar{t}_\alpha - \tfrac{1}{2}x_\alpha x_\alpha \bar{t}_1)\mathrm{d}S + \frac{p}{I}k\right] \tag{22}$$

2) $\varphi_i (i = 1,2,3)$ or Q, and W or N prescribed.
When φ_i is prescribed, say with $i = 1$, one obtains

$$U_{,3}(x_3^0) + \epsilon^2\chi_{,1}(0,0)U_{,333}(x_3^0) = -\varphi_1(x_3^0) \tag{23}$$

The remainder of this section is devoted to a consideration of the order of the beam theory developed here.

First, in the preceding derivation, only the outer expansion was developed and employed. The inner expansion, which provides the exact boundary conditions for the outer expansion, is beyond engineering scope. Thus, when the displacement components are prescribed at the ends of the beam, all engineering beam theories exhibit an asymptotic error in the zeroth order. A similar discussion was given by Gregory and Wan,[14,15] for plate aymptotic expansions.

Second, in the outer expansion, all displacements can be calculated to second-order corrections from asymptotic expansions Eqs. (6), i.e.,

$$v_1 = U(x_3) + \epsilon^2 \frac{\nu}{2}(x_1^2 - x_2^2)U_{,33}(x_3) \tag{24a}$$

$$v_2 = \epsilon^2 \nu x_1x_2U_{,33}(x_3) \tag{24b}$$

$$v_3 = \epsilon(-x_1U_{,3}(x_3) + \epsilon^2\{[\chi(x_1,x_2) - \chi(0,0)] \times U_{,333}(x_3) + W(x_3)\}) \tag{24c}$$

The stress components, however, can only be calculated to the lowest order. The determination of the $\tau_{\alpha\beta}$ is a two-dimensional problem that is not in the purview of engineering beam theory. Only for some special case, say, the plane stress beam considered by Rehfield and Murthy,[6] can one obtain an analytical form for the $\tau_{\alpha\beta}^{(0)}$. The present authors are developing such expressions for the case of thin-walled beams where the $\tau_{\alpha\beta}^{(0)}$ can be obtained by employing thin-shell assumptions.

As a result, the $\tau_{3\alpha}$ are calculated to the zeroth order only because the next order correction needs the results for the $\tau_{\alpha\beta}$. The expressions for the $\tau_{3\alpha}$ are

$$\tau_{31} = \frac{\epsilon^2}{2(1+\nu)}\left[\frac{\nu}{2}(x_1^2 - x_2^2) + \chi_{,1}\right]U_{,333}(x_3) \tag{25a}$$

$$\tau_{32} = \frac{\epsilon^2}{2(1+\nu)}(\nu x_1x_2 + \chi_{,2})U_{,333}(x_3) \tag{25b}$$

Stress τ_{33} can be calculated using the full term of the zeroth order and part of the second order, i.e.,

$$\tau_{33} = \epsilon(-x_1U_{,33}(x_3) + \epsilon^2\{[\chi(x_1,x_2) - \chi(0,0)](p/I) + W(x_3) + \nu\tau_{\alpha\alpha}\}) \tag{26}$$

where $\tau_{\alpha\alpha}^{(0)}$ are not exactly known for the arbitrary cross sections. Some approximation needs to be made in this case, such as

$$\tau_{\alpha\alpha}^{(0)} = K_0 + K_1x_1 + K_2x_2$$

The constants K_0, K_1, and K_2 are determined by Eqs. (18), which give a global correctness to the stress $\tau_{\alpha\alpha}^{(0)}$, but not a pointwise one.

IV. Discussion and Comparison

An engineering beam theory is derived based upon asymptotic expansions. Neither a priori static nor kinematic assumptions are made. Retaining the features of a one-dimensional beam theory, it can be used as easily as the classical Timoshenko beam theory. When $p(x_3) = 0$ and with the use of the appropriate end boundary conditions, this beam theory reduces to the St. Venant bending and flexure solutions. Some essential phenomena show up in the present formulation [for example, the axial force N induced by lateral loadings $p(x_3)$] that are not present in the Timoshenko theory. It is to be noted that the presence of N is not a nonlinear consideration as linear theory is employed here. Rather, as also shown by Rehfield and Murthy,[6] the N comes from the higher-order cross-section deformation. From a comparison with the given asymptotic expansion, we see that the present theory contains more terms of second order than does Timoshenko's theory. It can thus be assumed that the present theory can yield a more accurate determination of the deflection of a beam in most cases. To obtain some quantitative comparison, we employ Cowper's formulation of Timoshenko's beam theory. Consistent with the notation used here, Cowper's formulas are

$$U_c(x_3) = \frac{1}{A}\int_S v_1 \mathrm{d}A, \qquad \varphi_2 = \frac{1}{\epsilon I}\int_S x_1v_3\mathrm{d}A$$

$$IU_{c,33}(x_3) + M = \epsilon^2 \frac{p}{I}k_c \tag{27}$$

$$M_{,3} = Q, \qquad Q_{,3} = -p \tag{28}$$

The constant k_c is the shear coefficient in his formulation, and it is given by

$$k_c = -\frac{\nu I(I_2 - I)}{2A} + \int_S x_1\chi(x_1,x_2)\mathrm{d}A \tag{29}$$

where $I_2 = \int_S x_2^2 \mathrm{d}A$. Subscript c on the preceding quantities indicates Cowper's definitions that are different from those of the present paper.

Comparing Eq. (29) with (20b), it is clear that Timoshenko's beam theory does not contain the integral term in Eq. (19) that is of the same order as the shear deformation term. In addition to this, the definition of the various k is different. Calculations have been carried out for k_a and k_c for an elliptical cross section (see Fig. 1). The comparison is shown in Table 1. From our examination of the results, one

Table 1 Shear coefficients of elliptical cross section

$a/b(a=1)$	k^a	k_c	k_a
1/0.1	−0.0587	−0.0558	−0.0606
1/0.5	−0.292	−0.281	−0.297
1/1.0	−0.576	−0.576	−0.552
1/2.0	−1.06	−1.23	−0.767

[a]Where $k = \int_S x_1 \chi(x_1,x_2) \mathrm{d}A$.

can conclude that Cowper's definition of the shear coefficient is not reasonable, especially for $a/b < 1$. The kinematic variable U_c that incorporates both the deflection of the centerline and deformation of the cross section loses its clear physical meaning for narrow cross sections. As a result, it is suggested that even for the classical Timoshenko beam theory k_a as defined by Eq. (20) gives a more reasonable value of the shear coefficient than does k_c.

Recent work by Rehfield and Murthy[6] based on the elasticity solution under plane stress conditions may be taken as a special case of the present theory. Their numerical results may thus be considered as an example of the present theory. For the cross sections other than plane stress or strain, the present formulation predicts beam response more accurately than does the Timoshenko theory, in most cases. We do not make the latter statement too strongly since the present engineering theory contains a zeroth-order error introduced by the use of approximate displacement boundary conditions at the end of the beam. This error is of the same order as that present in the Timoshenko and Bernoulli-Euler beam theories. A discussion on the accuracy of Timoshenko beam theory by Nicholson and Simmonds[16] and later by many other active researchers in this field (see, for example, Fan[12]) may be applied to all engineering theories.

Appendix A

The auxiliary function $\chi(x_1,x_2)$ used in the previous sections is defined by

$$\chi_{,\alpha\alpha} = 2x_1 \qquad \text{in } S \tag{A1}$$

$$\chi_{,\alpha} n_\alpha = -\frac{\nu}{2}(x_1^2 - x_2^2)n_1 - \nu x_1 x_2 n_2 \qquad \text{on } \partial S \tag{A2}$$

For some cross sections, this function has an analytical form that makes it easy to be used in our formulations. All the solutions listed in this Appendix can be found in some standard elasticity textbooks (e.g., Sokolnikoff[13]).

1) Circle:

$$\chi = -(3/4 + \nu/2)a^2 x_1 + 1/4(x_1^3 - 3x_1x_2^3) + x_1x_2^2 \tag{A3}$$

where a is the radius of the circle.

2) Hollow circle:

$$\chi = -(3/4 + \nu/2)[(a^2+b^2)r + a^2b^2/r]\cos\theta + r^3/4 \cos 3\theta + r^3 \cos\theta \sin^2\theta \tag{A4}$$

where a and b are the outer and the inner radii, respectively.

3) Rectangle:

$$\chi = \left[-(1+\nu)a^2 + \frac{\nu b^2}{3}\right]x_1 + \frac{(2+\nu)}{6}(x_1^3 - 3x_1x_2^2) + \frac{4\nu b^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh(n\pi x_1/b)}{n^3\cosh(n\pi a/b)}\cos\left(\frac{n\pi x_2}{b}\right) + x_1x_2^2 \tag{A5}$$

where the length of the side parallel to the x_1 axis is $2a$ and the length of the side parallel to the x_2 axis is $2b$.

4) Ellipse:

$$\chi = -\frac{a^2[2(1+\nu)a^2 + b^2]}{3a^2+b^2}x_1 + \frac{2a^2 + b^2 + \nu(a^2-b^2)/2}{9a^2+3b^2}(x_1^3 - 3x_1x_2^2) + x_1x_2^2 \tag{A6}$$

where a is the semi-axis in the x_1 direction and b is the semi-axis in the x_2 direction.

Appendix B

Given here is a brief presentation of the application of the newly developed refined beam theory. The particular case considered is that of a simply supported beam subjected to a uniform loading along the x_3 axis (see Fig. B1).

For the loading being considered, the solution of Eq. (21b) is given by

$$U(x_3) = \frac{p}{24I}x_3^4 + \frac{A_2}{2}x_3^2 + A_0 \tag{B1}$$

Constants A_2 and A_0 are determined from the end conditions

$$M(x_3 = 1) = 0, \qquad U(x_3 = 1) = 0 \tag{B2}$$

The first of these conditions, upon use of relation Eq. (22), can be expressed as

$$U_{,33}(x_3 = 1) = \frac{\epsilon^2}{I}\left(\frac{\nu}{2} + \frac{k_a}{I}\right)p \tag{B3}$$

Satisfaction of these conditions yields

$$U(x_3) = \frac{p}{24I}x_3^4 + \frac{px_3^2}{2I}\left[-1/2 + \epsilon^2\left(\frac{\nu}{2} + \frac{k_a}{I}\right)\right] + \frac{p}{I}\left[\frac{5}{24} - \frac{\epsilon^2}{2}\left(\frac{\nu}{2} + \frac{k_a}{I}\right)\right] \tag{B4}$$

In the following let superscript d refer to dimensioned quantities. We then have the following relations:

$$U^d = \frac{UL\sigma}{E}, \qquad p^d = \frac{r^4}{l^3}\sigma p, \qquad I^d = IR^4 = \frac{\pi}{4}r^4 \tag{B5}$$

Solution Eq. (B4) can then be rewritten as

$$U^d(y_3) = \frac{p^d}{24EI^d}y_3^4 + \frac{p^d}{2EI^d}l^2y_3^2\left[-1/2 + \left(\frac{r}{l}\right)^2\left(\frac{\nu}{2} + \frac{k_a}{I}\right)\right] + \frac{p^d}{EI^d}l^4\left[\frac{5}{24} - 1/2\left(\frac{r}{l}\right)^2\left(\frac{\nu}{2} + \frac{k_a}{I}\right)\right] \tag{B6}$$

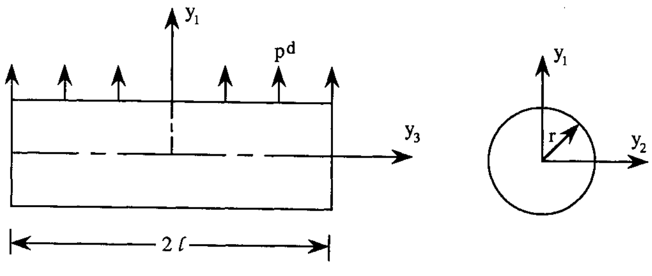

Fig. B1 Simply supported beam.

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